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SQUEEZED STATES OF A PARTICLE IN MAGNETIC FIELD

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For a charged particle in a homogeneous magnetic field, we construct stationary squeezed states which are eigenfunctions of the Hamiltonian and the non-Hermitian operator $\hat{X}_\Phi = \hat{X}\cos\Phi + \hat{Y}\sin\Phi$, $\hat{X}$ and $\hat{Y}$ being the coordinates of the Larmor circle center and Φ is a *complex* parameter. In the family of the squeezed states, the quantum uncertainty in the Larmor circle position is minimal. The wave functions of the squeezed states in the coordinate representation are found and their properties are discussed. Besides, for arbitrary gauge of the vector potential we derive the symmetry operators of translations and rotations.

The problem of a charged quantum particle moving on a plane subject to a homogeneous magnetic field is met in various physical contexts, and it has been extensively studied in the literature and presented in text-books [1]. A specific feature of the problem is that the energy spectrum given by discrete Landau levels is multiple degenerate: the number of linearly independent eigenstates belonging to the N-th Landau level is proportional to the area of the plane accessible to the particle. The degeneracy is related to the translational invariance: As a classical Larmor circle can be put anywhere in the plane, a suitably defined operator of translation $\hat{T}_{\boldsymbol{a}} = e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\boldsymbol{P}}}$, $\hat{\boldsymbol{P}}$ being the generator of translations, commutes with the Hamiltonian and upon acting on an energy eigenfunction $\psi_N(\boldsymbol{r})$ produces another eigenstate, shifted in space: $|\hat{T}_{\boldsymbol{a}}\psi_N(\boldsymbol{r})|^2 = |\psi_N(\boldsymbol{r}+\boldsymbol{a})|^2$ [2,3]. Non-collinear translations do not commute in a magnetic field, and $[\hat{P}_x, \hat{P}_y] \neq 0$. The existence of two non-commuting Hermitian operators, $\hat{P}_x$ and $\hat{P}_y$, each of them commuting with the Hamiltonian leads [1] to the degeneracy.

The stationary wave functions corresponding to a degenerate Landau level may be chosen to be eigenfunctions of either $\hat{P}_x$ or $\hat{P}_y$ (but not both simultaneously). The eigenvalues of $\hat{P}_{x,y}$ are real, and the translation operator $e^{ia\hat{P}_x}$ (or $e^{ia\hat{P}_y}$) applied to the corresponding eigenfunction gives only an overall phase factor. The modulus remains unchanged by the translations, so that the eigenstates must be infinitely extended in the $x-$ (or $y-$) direction. The wave functions of an electron in a magnetic field first found by Landau [1,4] give an example: factorized as $e^{ipx}\chi(y)$, they are eigenfunctions of $\hat{P}_x$ and are infinitely extended in the $x-$direction (strip-like states).

As discussed later, in the relation

$$\hat{\boldsymbol{P}} = \frac{e}{c}\boldsymbol{B} \times \hat{\boldsymbol{R}}, \tag{1}$$

$\hat{\boldsymbol{R}} = (\hat{X}, \hat{Y})$ has the meaning of the operator corresponding to the classical coordinate of the Larmor circle center (the guiding center); $\hat{X} = \frac{c}{eB}\hat{P}_y$ and $\hat{Y} = -\frac{c}{eB}\hat{P}_x$. The variable $\boldsymbol{R}$ has a simple classical interpretation, and for this reason, it will be used below rather than $\boldsymbol{P}$.

Instead of $\hat{X}$ or $\hat{Y}$ ($\hat{P}_x$ or $\hat{P}_y$), one may choose their Hermitian linear combination $\hat{X}_\Phi = \hat{X}\cos\Phi + \hat{Y}\sin\Phi$ with a real Φ. The corresponding eigenstates are "strips" the orientation of which depends on the angle Φ. A different class of states can be obtained if the wave function is chosen to be an eigenfunction of the *non-Hermitian* operator $\hat{X}_\Phi$ with a *complex* "angle" $\Phi = \Phi_1 + i\Phi_2$. By virtue of the relation in Eq.(1), the eigenfunctions of $\hat{X}_\Phi$ are also eigenfunctions of $\hat{P}_{\Phi+\frac{\pi}{2}}$. In the case of a general complex Φ, eigenvalues of the non-Hermitian operator $\hat{P}_{\Phi+\frac{\pi}{2}}$ are complex numbers, and the above argument concerning an infinite extension of the state is not applicable; the eigenfunctions turn out to be localized (*i.e.* the wave function vanishes at infinity).

In the terminology of quantum optics (for a review see [5] and references therein) these states belong to the class of *squeezed* states, generalization of the coherent states. In optics the squeezed state is defined as an eigenfunction of a non-Hermitian operator, $\hat{x} - i\lambda\hat{p}$, built of two non-commuting variables, the coordinate and momentum of a harmonic oscillator (λ being the squeeze parameter). A distinctive feature of squeezed states is that the quantum uncertainty in the non-commuting variables, is as minimal as allowed by the uncertainty relation (minimum uncertainty states). The purpose of the paper to analyze properties of the squeezed states, eigenfunctions of $\hat{X}_\Phi$.

Solutions to the Schrödinger equation for a charge in a magnetic field corresponding to non-spreading wave packets with a classical dynamics – the coherent states in the modern terminology – were first built by Darwin as early as in

1928 [6]. More recently, the coherent states in the magnetic field problem have been extensively studied by Malkin and Man'ko [7], and Feldman and Kahn [8] (see also Ref. [9–11]). In the coherent states, the quantum uncertainties in the X and Y coordinates of the Larmor center are equal. Various generalizations to the squeezed states have been done by Dodonov *et al.* [12,13] and Aragone [14]: In the general squeezed state, the uncertainty in one of the coordinates is reduced at the expense of the other one so that their product remains intact.

In the present paper, we consider *stationary* states, building the squeezed states from the energy eigenfunction belonging to a given Landau level. Being stationary, these states are of different kind than the moving squeezed wave packets of Ref. [12,14].

The paper is organized as follows. In Section I, we review some general features of the quantum problem of a charge in a magnetic filed. In Section II, we define the squeezed states and explicitly find the wave functions in the coordinate representation. In Section III, properties of the squeezed states are analyzed. In the Appendix, we suggest a method which allows one to construct symmetry operators for an arbitrary gauge of the vector potential and apply the method to the case of a homogeneous magnetic field.

I. GENERAL PROPERTIES

The Hamiltonian of a particle with mass m and electric charge e moving in the $x-y$ plane in a magnetic field reads

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right)^2 , \tag{2}$$

here the vector potential $\boldsymbol{A}(A_x, A_y)$ corresponds to a homogeneous magnetic field perpendicular to the plane, $(\mathbf{rot}\ \boldsymbol{A})_z = B$. The choice of signs in some of the below formulae depends on the sign of eB; for definiteness, we assume $eB > 0$. In terms of the ladder operators

$$\hat{c} = \frac{1}{\sqrt{2}\, l\, \omega_c}\left(\hat{v}_x + i\hat{v}_y\right) \quad , \quad \hat{c}^\dagger = \frac{1}{\sqrt{2}\, l\, \omega_c}\left(\hat{v}_x - i\hat{v}_y\right) \tag{3}$$

where $\hat{v}_{x,y}$ are the non-commuting components of the velocity operator

$$m\hat{\boldsymbol{v}} = \hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A} \quad , \quad [\hat{v}_x, \hat{v}_y] = i\frac{\hbar^2}{m^2 l^2} , \tag{4}$$

(the cyclotron frequency $\omega_c = \frac{|eB|}{mc}$ and the magnetic length $l = \sqrt{\hbar c/|eB|}$), the Hamiltonian Eq.(2) can be conveniently written as

$$\hat{H} = \hbar\omega_c\left(\hat{c}^\dagger\hat{c} + \frac{1}{2}\right) \quad , \quad \left[\hat{c}, \hat{c}^\dagger\right] = 1 . \tag{5}$$

In the presence of a homogeneous magnetic field, the translations in the x-y plane and rotations around the $z-$axis remain symmetry elements. The reflection, σ_v, in a plane passing through the $z-$axis (yOz, for definiteness), reverses the magnetic field and is not a symmetry transformation. However, the product $\sigma_T \equiv T \cdot \sigma_v$ of time-reversal T and σ_v , both reversing the field, is a valid symmetry.

It is well-known that the Hamiltonian Eq.(2) may not commute with the operators associated with the physical symmetries because the vector field $\boldsymbol{A}(\boldsymbol{r})$ has a lower symmetry than the corresponding magnetic field. If this is the case, the coordinate transformation should be accompanied by a certain gauge transformation which compensates the change in $\boldsymbol{A}(\boldsymbol{r})$ [3]. A procedure of constructing the transformation, which is valid for arbitrary gauge of the vector potential, is presented in the Appendix.

As shown in the Appendix, the operators of finite translations, $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{a}$, are built of the generator of translations

$$\hat{\boldsymbol{P}} = \hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A} + \frac{e}{c}\boldsymbol{B} \times \boldsymbol{r}. \tag{6}$$

The operator $\hat{\boldsymbol{P}}$ commutes with the Hamiltonian Eq.(2) as manifestation of the translation invariance preserved in a homogeneous magnetic field. The components of $\hat{\boldsymbol{P}}$ obey the commutation relation

$$[\hat{P}_x, \hat{P}_y] = -i\frac{e}{c}\hbar B . \tag{7}$$

Equation (6) is valid in an arbitrary gauge of the vector potential $\boldsymbol{A}$. In case of the symmetric gauge, $\boldsymbol{A}=\frac{1}{2}(\boldsymbol{B}\times\boldsymbol{r})$, Eq.(6) gives the expression first found by Zak [2].

Presenting $\hat{\boldsymbol{P}}$ in the form in Eq.(1), one indeed recognizes in $\hat{\boldsymbol{R}}$ the center of the Larmor circle (the guiding center),

$$\hat{\boldsymbol{R}}=\hat{\boldsymbol{r}}+\frac{mc}{eB^2}\hat{\boldsymbol{v}}\times\boldsymbol{B}, \tag{8}$$

an integral of motion known from classical mechanics. The following commutation relations

$$\left[\left(\hat{\boldsymbol{R}}\right)_l,\ \left(\hat{\boldsymbol{P}}\right)_m\right]=i\hbar\delta_{lm} \tag{9}$$

$$[\hat{X},\hat{Y}]=\frac{l^2}{i} \tag{10}$$

can be readily derived from Eqs.(7), and (8).

II. SQUEEZED STATES

From Eq.(10), the X- and Y-coordinates of the Larmor circle center are incompatible quantum variables. Given the commutator Eq.(10), their variances obey the standard (see e.g. [15]) uncertainty relation:

$$(\Delta X)^2(\Delta Y)^2\geq\frac{1}{4}l^4\ ,$$

where the variance of a variable A is defined as $(\Delta A)^2=\left\langle\left(\Delta\hat{A}\right)^2\right\rangle$ with $\Delta\hat{A}$ here and below standing for $\Delta\hat{A}=\hat{A}-\left\langle\hat{A}\right\rangle$. If the uncertainty relation is satisfied with the equality sign, it is said that the system is in a minimum uncertainty state or, in other words, in a coherent or, more generally, in a squeezed state.

We construct the stationary *squeezed* state, $|\boldsymbol{R},N\rangle$ as a simultaneous eigenfunction of the Hamiltonian Eq.(2) and the operator $\hat{X}_\Phi$,

$$\hat{X}_\Phi\equiv\hat{X}\cos\Phi+\hat{Y}\sin\Phi\ , \tag{11}$$

where Φ is a complex parameter $\Phi=\Phi_1+i\Phi_2$. Under a ϕ-rotation around the z-axis, Φ transforms to $\Phi\to\Phi'=\Phi-\phi$, and $\hat{X}_\Phi$ can be also represented as

$$\hat{X}_\Phi=\hat{X}'\cosh\Phi_2+i\hat{Y}'\sinh\Phi_2 \tag{12}$$

where $\hat{X}'=\hat{X}\cos\Phi_1+\hat{Y}\sin\Phi_1$ and $\hat{Y}'=-\hat{X}\sin\Phi_1+\hat{Y}\cos\Phi_1$ are the Cartesian components of the guiding center $\boldsymbol{R}$ in the principal axes where $\Phi_1'=0$.

The state $|\boldsymbol{R},N\rangle$ is found from the following system of equations

$$\hat{H}|\boldsymbol{R},N\rangle=\hbar\omega_c(N+\frac{1}{2})|\boldsymbol{R},N\rangle\ , \tag{13}$$

$$\hat{X}_\Phi|\boldsymbol{R},N\rangle=(X\cos\Phi+Y\sin\Phi)|\boldsymbol{R},N\rangle\ . \tag{14}$$

The quantum numbers of a state are the Landau level number N, and the expectation value of the guiding center position, $\boldsymbol{R}(X,Y)$; two real parameters X and Y specify the complex eigenvalue X_Φ.

The operator $\hat{X}_\Phi$ is not Hermitian and the eigenvalue problem may or may not have solutions among physically admissible normalizable functions, $\langle\boldsymbol{R},N|\boldsymbol{R},N\rangle=1$. To find necessary conditions for the existence of physical solutions, we note that Eq.(14) leads to $\langle\boldsymbol{R},N|\Delta\hat{X}_\Phi^\dagger\Delta\hat{X}_\Phi|\boldsymbol{R},N\rangle=0$, or, using the representation in Eq.(12) and Eq.(10),

$$(\Delta X')^2+\tanh^2\Phi_2(\Delta Y')^2=-l^2\tanh\Phi_2\ . \tag{15}$$

Observing that the l.h.s. is positive definite, we conclude that Eq.(14) has normalizable solutions only if $\Phi_2<0$ (or, more generally, $eB\Phi_2<0$).

The real and imaginary parts of the relation $\langle\boldsymbol{R},N|\left(\Delta\hat{X}_\Phi\right)^2|\boldsymbol{R},N\rangle=0$, give

$$(\Delta Y')^2 \, \tanh^2 \Phi_2 = (\Delta X')^2 \tag{16}$$

$$\left\langle \Delta \hat{X}' \Delta \hat{Y}' + \Delta \hat{Y}' \Delta \hat{X}' \right\rangle = 0 \tag{17}$$

$$(\Delta X')^2 (\Delta Y')^2 = \frac{l^2}{4} \ . \tag{18}$$

The last relation follows from the first two combined with Eq.(15). Also,

$$(\Delta X')^2 = \frac{l^2}{2} |\tanh \Phi_2| \ , \ (\Delta Y')^2 = \frac{l^2}{2} |\coth \Phi_2| \ . \tag{19}$$

According to Eq.(18), the eigenfunctions of $\hat{X}_\Phi$ indeed belong to the class of minimum uncertainty states. From Eq.(17), one sees that the physical meaning of Φ_1 is to show the orientation of the principal axes, along which the quantum fluctuations of the guiding center position are independent. It follows from Eq.(16) that Φ_2 controls the relative uncertainty of the projection of the guiding center onto the principal axes.

To find the wave functions of the squeezed states, we first consider the states from the Landau level $N = 0$, $|\boldsymbol{R}, 0\rangle$ and solve the following system of equations

$$\hat{c}\, |\boldsymbol{R}, 0\rangle = 0 \ , \tag{20}$$

$$\hat{X}_\Phi |\boldsymbol{R}, 0\rangle = X_\Phi |\boldsymbol{R}, 0\rangle \ . \tag{21}$$

(Eq.(20) is equivalent to Eq.(13) for the ground state $N = 0$.) To find the explicit form of the wave functions, we choose the symmetric gauge

$$\boldsymbol{A} = \frac{1}{2} \, \boldsymbol{B} \times \boldsymbol{r} \, , \tag{22}$$

where different directions are treated on equal footing.

In the notations

$$\begin{aligned} \tilde{x}_\Phi &= (x - X) \cos \Phi + (y - Y) \sin \Phi \ , \\ \tilde{y}_\Phi &= -(x - X) \sin \Phi + (y - Y) \cos \Phi \ , \end{aligned} \tag{23}$$

($\tilde{y}_\Phi = \tilde{x}_{\Phi + \frac{\pi}{2}}$), the operators in Eq.(20) and Eq.(21) take the form

$$i \frac{\sqrt{2}}{l} e^{-i\Phi} \hat{c} = \frac{\partial}{\partial \tilde{x}_\Phi} + i \frac{\partial}{\partial \tilde{y}_\Phi} + \frac{1}{2l^2} \big(\tilde{x}_\Phi + i \tilde{y}_\Phi \big) + \frac{1}{2l^2} \big(X_\Phi + i Y_\Phi \big) \tag{24}$$

$$\hat{X}_\Phi = \tilde{x}_\Phi - i l^2 \frac{\partial}{\partial \tilde{y}_\Phi} + X_\Phi \ , \tag{25}$$

where $Y_\Phi = -X \sin \Phi + Y \cos \Phi$.

In the coordinate representation, Eqs.(20,21) become a system of linear differential equations of the first order for $\Psi(\boldsymbol{r}|0, \boldsymbol{R}) = \langle \boldsymbol{r} | \boldsymbol{R}, 0 \rangle$. The normalized solution reads

$$\Psi(\boldsymbol{r} \mid 0, \boldsymbol{R}) = C_\Phi \exp \left(-\frac{1}{2l^2} \left(\tilde{x}_\Phi^2 + i \tilde{x}_\Phi \tilde{y}_\Phi + i \tilde{x}_\Phi Y_\Phi - i \tilde{y}_\Phi X_\Phi \right) \right), \tag{26}$$

where C_Φ is the normalization constant

$$|C_\Phi|^2 = \frac{1}{2\pi l^2} \sqrt{1 - \exp\, 2i(\Phi^* - \Phi)} \ . \tag{27}$$

As expected, the function in Eq.(26) is normalizable and Eq.(27) is meaningful, only if $Im\, \Phi < 0$.

The normalized states for the N-th Landau level, eigenfunctions of the Hamiltonian in Eq.(5), can be now found from

$$\Psi(\boldsymbol{r} \mid N, \boldsymbol{R}) = \frac{1}{\sqrt{N!}} \big(\hat{c}^\dagger \big)^N \Psi(\boldsymbol{r} \mid 0, \boldsymbol{R}) \ . \tag{28}$$

The calculation can be easily done with the help of the following identity

$$
\begin{aligned}
\hat{c}^\dagger &= e^{-i\Phi} e^{\Lambda} \left(\frac{l\hbar}{i\sqrt{2}} \frac{\partial}{\partial \tilde{x}_\Phi} \right) e^{-\Lambda} \ , \\
\Lambda &\equiv \frac{1}{2l^2} \left(\tilde{x}_\Phi^2 - i\tilde{x}_\Phi \tilde{y}_\Phi - i\tilde{x}_\Phi Y_\Phi + i\tilde{y}_\Phi X_\Phi \right) \ .
\end{aligned} \tag{29}
$$

After some algebra we obtain,

$$
\Psi(\boldsymbol{r}|N,\boldsymbol{R}) = C_\Phi \frac{\left(ie^{-i\Phi}\right)^N}{\sqrt{2^N N!}} \exp\left(i\frac{e}{2\hbar c} \boldsymbol{B} \cdot (\boldsymbol{R} \times \boldsymbol{r}) - \frac{1}{2l^2} \tilde{x}_\Phi \left(\tilde{x}_\Phi + i\tilde{y}_\Phi \right) \right) H_N \left(\frac{\tilde{x}_\Phi}{l} \right) \tag{30}
$$

where $H_N(\xi)$ is the Hermite polynomial, $H_N(\xi) = (-1)^N e^{\xi^2} \frac{d^N e^{-\xi^2}}{d\xi^N}$. In the coordinate representation, this expression gives the wave function of the squeezed state $|\boldsymbol{R}, N\rangle$ centered at $\boldsymbol{R}$ and belonging to the N-th Landau level (in the symmetric gauge Eq.(22)); the "rotation angle" Φ in Eq.(23) is a complex parameter, $Im\,\Phi < 0$.

III. PROPERTIES

The basic features of squeezed states can be seen in Figs. 1 and 2, where the density and the current are plotted for a typical state: $N = 2$, $\Phi = i\Phi_2$, $|\tanh\Phi_2| = 0.1$. The squeezed state is *localized* in the sense that it has a finite extension in both $x-$ and $y-$directions. Qualitatively, the squeezed state is a superposition of classical Larmor orbits of radius $\rho_N = \sqrt{\frac{2E_N}{m\omega^2}}$, the centers of which are positioned in the vicinity of $\boldsymbol{R}$ with a typical deviation $\Delta X'$ and $\Delta Y'$. From Eq.(19), $\Delta X'/\Delta Y' = |\tanh\Phi_2| < 1$ so that the state is elongated in the direction of the principal y'-axis. When Φ_2 tends to zero, the elongation increases and the squeezed state asymptotically transforms into a "strip" (of length $\sim l/|\Phi_2|$).

The wave function of a squeezed state from the N-th Landau level has N *isolated zeroes* in the $x-y$ plane. The zeroes are at the points on the line $Im\ \tilde{x}_\Phi = 0$, where the Hermite polynomial $H_N(\frac{\tilde{x}_\Phi}{l})$ has its N roots. In the limit $\Phi_2 \to -\infty$, the zeroes gather together at the point $\boldsymbol{r} = \boldsymbol{R}$. This limit, where $\Delta X = \Delta Y = \frac{l}{2}$, gives the stationary coherent state introduced by Malkin and Man'ko [7], the angular momentum eigenstate with the eigenvalue $L_z = -\hbar N$. In the coherent states, the probability density is rotationally invariant.

In quantum optics, squeezed states can be presented as the result of the action of the "squeezing operator" on the coherent state. Similar to [12], squeezing of the cylindrically symmetric coherent states ($Im\,\Phi \to -\infty$) is achieved by applying

$$
\hat{S} = \exp\left\{ \frac{i}{2l^2} r \left(\hat{X}'\hat{Y}' + \hat{Y}'\hat{X}' \right) \right\} \ , \tag{31}
$$

where $\tanh r = e^{2\Phi_2}$. In optics there are certain non-linear processes with the evolution operator in the form of $\hat{S}$ [16]. A coherent state then evolves into the squeezed state. For the case of a particle in a magnetic field, the analogous problem of preparation of a squeezed state has been considered in [13].

As discussed in the Appendix, the product of the mirror and time reversal transformations is a valid symmetry element. As a consequence, the distribution of the density and current are mirror symmetric (relative to the principal axes) as also apparent in Figs.(1-4).

Within a given Landau level, the squeezed states, eigenfunctions of a non-Hermitian operator $\hat{X}_\Phi$, are *non-orthogonal*. For the states defined in Eq.(30) with the real positive normalization constant C_Φ Eq.(27), the overlap integral reads

$$
\begin{aligned}
\langle \boldsymbol{R}; \Phi, N \mid \boldsymbol{R}'; \Phi', N' \rangle &= \delta_{NN'} \frac{\Big(1 - \exp\big(2i(\Phi^* - \Phi)\big)\Big)^{1/4} \Big(1 - \exp\big(2i(\Phi'^* - \Phi')\big)\Big)^{1/4}}{\Big(1 - \exp\big(2i(\Phi^* - \Phi')\big)\Big)^{1/2}} \times \\
&\times \exp\left(\frac{ie}{2\hbar c} \boldsymbol{B} \cdot (\boldsymbol{R}' \times \boldsymbol{R}) + \frac{i}{2l^2} \frac{\left(X'_{\Phi'} - X_{\Phi'}\right)\left(X'_{\Phi^*} - X_{\Phi^*}\right)}{\sin\left(\Phi' - \Phi^*\right)} \right)
\end{aligned} \tag{32}
$$

($Re \Big(1 - \exp\big(2i(\Phi^* - \Phi')\big)\Big)^{1/2} > 0$). The overlap of the states differing in the Larmor center position $\boldsymbol{R}$ or the parameter Φ does not depend on the Landau level number N as it follows from the Eq.(28) and the commutation

relation in Eq.(5). Therefore, the overlap integral in Eq.(32) can be calculated using the Gaussian wave functions for $N = 0$ in Eq.(26).

Repeating the derivation known in the theory of coherent states of a harmonic oscillator (see e.g. [5]), one can show that the set of squeezed states is complete *i.e.* the closure relation,

$$\hat{1} = \sum_{N=0}^{\infty} \int \frac{d^2\boldsymbol{R}}{2\pi l^2} |\boldsymbol{R}, N\rangle\langle\boldsymbol{R}, N| \ , \tag{33}$$

is valid. As in the harmonic oscillator case, the states $|\boldsymbol{R}, N\rangle$ with continuously varying $\boldsymbol{R}$ form an overcomplete set within the N-th Landau level. Repeating Perelomov's arguments [17], one can show that the subset of states with $\boldsymbol{R}$'s on the sites of a periodic lattice is overcomplete if the lattice is too dense (the unit cell area $s_0 < 2\pi l^2$) and is not complete for a too dilute lattice ($s_0 > 2\pi l^2$). When $s_0 = 2\pi l^2$ (*i.e.* the flux through the unit cell equals to the flux quantum $\frac{hc}{e}$), the system of the functions is complete and it remains complete even if a *single* state is removed; it becomes incomplete, however, if any *two* states are removed.

In conclusion, stationary squeezed states of a charged particle in a homogeneous magnetic field have been constructed and analyzed. The distinctive feature of the squeezed states is the minimal quantum uncertainty of the position of the Larmor circle center. The family of the squeezed states is characterized by the squeezing parameter $Im\,\Phi$ variation of which allows one to transform gradually strip-like states (infinitely extended in the direction controlled by $Re\,\Phi$) to eigenfunctions of the angular momentum with rotationally invariant density distribution. The squeezed states have a rather simple structure: As it follows from Eqs.(30 and 23), a general squeezed state can be obtained from a Landau "strip" by a complex angle rotation of the coordinates. The simplicity of the construction gives the hope that the squeezed states may turn out to be useful.

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APPENDIX A: SYMMETRIES OPERATORS

The physical symmetry of a system in an external magnetic field is controlled by the symmetry of the vector field $\boldsymbol{B}(\boldsymbol{r})$ (among other factors). However, the Hamiltonian, $\hat{H} = \hat{H}_{\boldsymbol{A}}$ contains the vector potential $\boldsymbol{A}(\boldsymbol{r})$ rather than $\boldsymbol{B}$. The vector field $\boldsymbol{A}(\boldsymbol{r})$ has a lower symmetry and, moreover, the spatial symmetry of $\boldsymbol{A}(\boldsymbol{r})$ is gauge dependent. For this reason, the Hamiltonian often does not commute with the operator corresponding to a physical symmetry element. A homogeneous magnetic field gives a simple example: the translation invariance is preserved but the vector potential is always $\boldsymbol{r}$-dependent. It is well-known [3] that the symmetry operator should include a certain gauge transformation which compensates the change of the $\boldsymbol{A}$-field generated by the symmetry transformation.

To build the modified operators on a regular basis, we suggest the following procedure. For any given symmetry element $\mathcal{O}$, it is always possible to find field $\boldsymbol{A}^{(\mathcal{O})}(\boldsymbol{r})$, $\textbf{rot}\ \boldsymbol{A}^{(\mathcal{O})} = \boldsymbol{B}$, which is invariant relative to $\mathcal{O}$: $\mathcal{O}\boldsymbol{A}^{(\mathcal{O})} = \boldsymbol{A}^{(\mathcal{O})}$. Whatever gauge is chosen for $\boldsymbol{A}$ in Eq.(2), the gauge transformation, $\hat{G}_{\mathcal{O}}^{-1}\hat{H}_{\boldsymbol{A}}\hat{G}_{\mathcal{O}}$, specified by

$$\hat{G}_{\mathcal{O}} = e^{i\chi^{(\mathcal{O})}} \ \ , \ \ \chi^{(\mathcal{O})}(\boldsymbol{r}) = \frac{e}{\hbar c}\int_0^{\boldsymbol{r}} d\boldsymbol{r}\left(\boldsymbol{A} - \boldsymbol{A}^{(\mathcal{O})}\right) \ \ , \tag{A1}$$

changes the vector potential entering the Hamiltonian from $\boldsymbol{A}$ to $\boldsymbol{A} - \boldsymbol{\nabla}_{\boldsymbol{r}}\chi^{(\mathcal{O})} = \boldsymbol{A}^{(\mathcal{O})}$, *i.e.* $\hat{G}_{\mathcal{O}}^{-1}\hat{H}_{\boldsymbol{A}}\hat{G}_{\mathcal{O}} = \hat{H}_{\boldsymbol{A}^{(\mathcal{O})}}$. By construction, $\mathcal{O}$ does not change the vector field $\boldsymbol{A}^{(\mathcal{O})}(\boldsymbol{r})$ and, therefore, the transformed operator $\hat{G}_{\mathcal{O}}^{-1}\hat{H}_{\boldsymbol{A}}\hat{G}_{\mathcal{O}}$ commutes with $\hat{\mathcal{O}}$, *i.e.*

$$\left(\hat{G}_{\mathcal{O}}^{-1}\hat{H}_{\boldsymbol{A}}\hat{G}_{\mathcal{O}}\right)\hat{\mathcal{O}} = \hat{\mathcal{O}}\left(\hat{G}_{\mathcal{O}}^{-1}\hat{H}_{\boldsymbol{A}}\hat{G}_{\mathcal{O}}\right) \quad \text{or} \quad \hat{H}_{\boldsymbol{A}}\left(\hat{G}_{\mathcal{O}}\hat{\mathcal{O}}\hat{G}_{\mathcal{O}}^{-1}\right) = \left(\hat{G}_{\mathcal{O}}\hat{\mathcal{O}}\hat{G}_{\mathcal{O}}^{-1}\right)\hat{H}_{\boldsymbol{A}} \ .$$

Therefore, it is the operator

$$\hat{\mathcal{O}}_{\boldsymbol{A}} = \hat{G}_{\mathcal{O}}\hat{\mathcal{O}}\hat{G}_{\mathcal{O}}^{-1} \ \ , \tag{A2}$$

which commutes with the Hamiltonian $\hat{H}_{\boldsymbol{A}}$ and represents the symmetry element $\mathcal{O}$. As such, the operator $\hat{\mathcal{O}}_{\boldsymbol{A}}$ depends on the gauge chosen for the vector potential $\boldsymbol{A}$ in the Hamiltonian, but its matrix elements are gauge invariant if the sandwiching functions are gauge transformed in the usual manner.

Equivalently, the symmetry operator in Eq.(A2) can be written as

$$\hat{\mathcal{O}}_{\boldsymbol{A}} = e^{i\left(\chi^{(\mathcal{O})}(\boldsymbol{r})-\chi^{(\mathcal{O})}(\hat{\mathcal{O}}\boldsymbol{r})\right)}\ \hat{\mathcal{O}} \tag{A3}$$

$$= \hat{\mathcal{O}}\ e^{i\left(\chi^{(\mathcal{O})}(\hat{\mathcal{O}}^{-1}\boldsymbol{r})-\chi^{(\mathcal{O})}(\boldsymbol{r})\right)}\,. \tag{A4}$$

Below, we analyze only the case of a homogeneous magnetic field but the method is generally applicable.

In the presence of a homogeneous magnetic field, the translations in the x-y plane and rotations around the $z-$axis remain (continuous) symmetry elements. As the reflection, σ_v, in a plane passing through the $z-$axis (yOz, for definiteness), reverses the magnetic field, σ_v is not a symmetry element. We note here that the product of time-reversal T and σ_v , both reversing the field, is a symmetry element. We denote the product by $\sigma_T \equiv T\cdot\sigma_v$.

The transformation $\mathcal{O}$ changes the wave function as, $\psi \xrightarrow{\mathcal{O}} \hat{\mathcal{O}}\psi$, where $\hat{\mathcal{O}}$ denotes the operator associated with $\mathcal{O}$. For translations and rotations,the form of the operator $\hat{\mathcal{O}}$ is obvious; in the case of σ_T,

$$\hat{\sigma_T}\psi(x,y) = \psi^*(-x,y)\,, \tag{A5}$$

and $\hat{\sigma_T}$ is an anti-linear anti-unitary operator. From $\hat{\mathcal{O}}^{-1}\hat{H}_{\boldsymbol{A}}\hat{\mathcal{O}} \equiv \hat{H}_{\mathcal{O}\boldsymbol{A}}$, the transformed vector field $\mathcal{O}\boldsymbol{A}$ can be found. Again, the result is obvious in case of rotations and translations. Under σ_T, which is the mirror reflection of the polar vector field $\boldsymbol{A}(\boldsymbol{r})$ in combination with time reversal ($\boldsymbol{A}\to-\boldsymbol{A}$), the vector potential transforms as: $\hat{\sigma_T}A_x(x,y) = A_x(-x,y)$, $\hat{\sigma_T}A_y(x,y) = -A_y(-x,y)$.

First, we consider the operator of a finite translation $\hat{\mathcal{O}} = \hat{T}_{\boldsymbol{a}}$, $\hat{T}_{\boldsymbol{a}}\psi(x,y) = \psi(x+a_x,y+a_y)$, $\boldsymbol{a}$ being the translation vector. The vector potential, $\boldsymbol{A}^{(\mathcal{O})} \equiv \boldsymbol{A}^{(\boldsymbol{a})}$, invariant relative to the translation along $\boldsymbol{a}$, $\boldsymbol{A}^{(\boldsymbol{a})}(\boldsymbol{r}) = \boldsymbol{A}^{(\boldsymbol{a})}(\boldsymbol{r}+\boldsymbol{a})$, can be taken as,

$$\boldsymbol{A}^{(\boldsymbol{a})} = \boldsymbol{n}\Big((\boldsymbol{B}\times\boldsymbol{r})\cdot\boldsymbol{n}\Big)\ \ ,\ \ \boldsymbol{a} = |\boldsymbol{a}|\boldsymbol{n}\ \ , \tag{A6}$$

("Landau gauge"). Using Eq.(A3), the operator of magnetic translation reads

$$\hat{T}^{(\boldsymbol{a})}_{\boldsymbol{A}} = e^{i\frac{e}{c\hbar}\int\limits_{\boldsymbol{r}+\boldsymbol{a}}^{\boldsymbol{r}} d\boldsymbol{r}'\left(\boldsymbol{A}(\boldsymbol{r}')-\boldsymbol{A}^{(\boldsymbol{a})}(\boldsymbol{r}')\right)}\, e^{\frac{i}{\hbar}\hat{\boldsymbol{p}}\cdot\boldsymbol{a}}, \tag{A7}$$

$\hat{\boldsymbol{p}} = \frac{\hbar}{i}\boldsymbol{\nabla}$ being the canonical momentum. As a consequence of the physical translational invariance, this operator commutes with the Hamiltonian Eq.(2) for arbitrary gauge of the vector potential $\boldsymbol{A}(\boldsymbol{r})$.

Up to terms linear in $\boldsymbol{a}$, $\hat{T}^{(\boldsymbol{a})}_{\boldsymbol{A}} = 1+\frac{i}{\hbar}\boldsymbol{a}\cdot\left(\hat{\boldsymbol{p}}-\frac{e}{c}\boldsymbol{A}+\frac{e}{c}\boldsymbol{A}^{(\boldsymbol{a})}\right)$ or, using Eq.(A6), $\hat{T}^{(\boldsymbol{a})}_{\boldsymbol{A}} = 1+\frac{i}{\hbar}\boldsymbol{a}\cdot\left(\hat{\boldsymbol{p}}-\frac{e}{c}\boldsymbol{A}+\frac{e}{c}\boldsymbol{B}\times\boldsymbol{r}\right)$. From here, one reads off the expression for the generator of translations $\hat{\boldsymbol{P}}$ and derives Eq.(6).

This derivation of Eq.(6) links $\hat{\boldsymbol{P}}$ to the translational symmetry. The same expression for $\hat{\boldsymbol{P}}$ can be derived in a more intuitive manner: First, one considers the operator of the guiding center $\hat{\boldsymbol{R}}$, the expression for which in Eq.(8) can be guessed from the correspondence principle. Since $\boldsymbol{R}$ is a classical integral of motion, $\hat{\boldsymbol{R}}$ must commute with the Hamiltonian. Now, one *defines* $\hat{\boldsymbol{P}}$ by Eq.(1) and comes immediately to Eq.(6).

Next we consider rotations *i.e.* $\mathcal{O} = R$. The "bare" operator of a rotation around the z-axis is $\hat{R} = e^{\frac{i}{\hbar}(\boldsymbol{r}\times\hat{\boldsymbol{p}})_z\phi}$, ϕ being the angle of the rotation. The $R-$invariant vector potential is $\boldsymbol{A}^{(R)} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$. Applying Eq.(A2), the symmetry operator reads:

$$\hat{R}_{\boldsymbol{A}}(\phi) = e^{i\frac{e}{c\hbar}\int\limits_0^{\boldsymbol{r}} d\boldsymbol{r}\left(\boldsymbol{A}-\boldsymbol{A}^{(R)}\right)}\, e^{\frac{i}{\hbar}(\boldsymbol{r}\times\hat{\boldsymbol{p}})_z\phi}\, e^{-i\frac{e}{c\hbar}\int\limits_0^{\boldsymbol{r}} d\boldsymbol{r}\left(\boldsymbol{A}-\boldsymbol{A}^{(R)}\right)}\,. \tag{A8}$$

In the limit $\phi\to 0$, $\hat{R}_{\boldsymbol{A}}(\phi) \approx 1+\frac{i}{\hbar}\hat{\mathcal{L}}_z\phi$, where the generator of rotations $\hat{\mathcal{L}}_z$ is

$$\hat{\mathcal{L}}_z = e^{i\frac{e}{c\hbar}\int\limits_0^{\boldsymbol{r}} d\boldsymbol{r}\left(\boldsymbol{A}-\boldsymbol{A}^{(R)}\right)}\, (\boldsymbol{r}\times\hat{\boldsymbol{p}})_z\, e^{-i\frac{e}{c\hbar}\int\limits_0^{\boldsymbol{r}} d\boldsymbol{r}\left(\boldsymbol{A}-\boldsymbol{A}^{(R)}\right)}\,.$$

Simplifying and using the expression for $\boldsymbol{A}^{(R)}$, one gets

$$\hat{\mathcal{L}}_z = \Big(\boldsymbol{r}\times\Big(\hat{\boldsymbol{p}}-\frac{e}{c}\boldsymbol{A}\Big)\Big)_z + \frac{e}{2c}\big(\boldsymbol{r}\times(\boldsymbol{B}\times\boldsymbol{r})\big)_z\ \ . \tag{A9}$$

For the case of a homogeneous magnetic field, this operator commutes with the Hamiltonian for arbitrary gauge of the vector potential and reduces to the usual angular momentum $\hat{L}_z = \boldsymbol{r}\times\hat{\boldsymbol{p}}$ when $\boldsymbol{A}, \boldsymbol{B} \to 0$. Also, $\hat{\mathcal{L}}_z = \hat{L}_z$ in the symmetric gauge $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$.

The operator $\mathcal{L}_z$ can be written in the following identically equivalent forms

$$\hat{\mathcal{L}}_z = \frac{1}{2}\frac{eB}{c}\hat{\boldsymbol{R}}^2 - \frac{mc}{eB}\hat{H} \tag{A10}$$

$$\hat{\mathcal{L}}_z = \frac{1}{2}\frac{c}{eB}\hat{\boldsymbol{P}}^2 - \frac{mc}{eB}\hat{H}; \tag{A11}$$

$\hat{H}$, and the (two-dimensional) vectors $\hat{\boldsymbol{P}}$ and $\hat{\boldsymbol{R}}$ are defined in Eqs.(2), (6) and (8), respectively. One sees that the integral of motion $\mathcal{L}_z$ is, actually, a function of the other conserving quantities $\boldsymbol{P}$ (or $\boldsymbol{R}$) and the energy H.

If the axis of the a rotation is shifted from the origin to the point $\boldsymbol{R}_0$, the generator of the rotations denoted as $\hat{\mathcal{L}}_{\boldsymbol{R}_0,z}$ reads

$$\hat{\mathcal{L}}_{\boldsymbol{R}_0,z} = \frac{1}{2}\frac{eB}{c}\left(\hat{\boldsymbol{R}} - \boldsymbol{R}_0\right)^2 - \frac{mc}{eB}\hat{H}\,. \tag{A12}$$

Note that the vector potential enters $\mathcal{L}_z$ Eq.(A9) and the generator of translations $\hat{\boldsymbol{P}}$ Eq.(6) only in the gauge covariant combination $\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}$ so that their matrix elements are gauge invariant.

Finally, we consider the combined mirror and time-reversal transformation in Eq.(A5): $\mathcal{O} = \sigma_T$. One can check that $\hat{\sigma}_T$ does not change the Hamiltonian in Eq.(2) if the symmetric gauge Eq.(22) is chosen. Ultimately, this is the reason for the mirror symmetry in the distribution of the density and current seen in Figs(1– 4).

[1] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon Press, New York, 1977).
[2] J. Zak, Phys. Rev. **136**, A776 (1964).
[3] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, part 2*, Pergamon Press (1980).
[4] L.D. Landau, Z.Physik **64**, 629 (1930).
[5] W.M. Zhang, D.H. Feng, R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[6] G.C. Darwin, Proc. Roy. Soc. (London) **117**, 258 (1928).
[7] I.A. Malkin and V.I. Man'ko, Zh. Eksp. Teor. Fiz. **55**,1014 (1968) [Sov. Phys. JETP **28**, 527 (1969)].
[8] A. Feldman and A.H. Kahn, Phys. Rev. **B 1**, 4584 (1970).
[9] S.T. Pavlov and A.V. Prokhorov, Fiz. Tver. Tela **32**, 3451 (1990) [Sov. Phys. Solid State **32**, 2001 (1991)].
[10] E.I. Rashba, L.E. Zhukov and A.L. Efros, Phys. Rev. B **55**, 5306 (1997).
[11] S. Varró, J.Phys. A: Math. Gen. **17**, 1631 (1984).
[12] V.V. Dodonov, E.V. Kurmyshev and V.I. Man'ko, in: Proceedings of the Lebedev Physics Institute, Vol. 176. Classical and quantum effects in electrodynamics, (Nova Science, Commack, NY, 1988) p.169.
[13] V.V. Dodonov, V.I. Man'ko, P.G. Polynkin, Phys. Lett. A **188**, 232 (1994).
[14] C. Aragone, Phys. Lett. A **175**, 377 (1993).
[15] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1965).
[16] D.F. Walls, G.J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994) p.137.
[17] A.M. Perelomov, Teor. Mat. Fiz. **6**, 213 (1971).

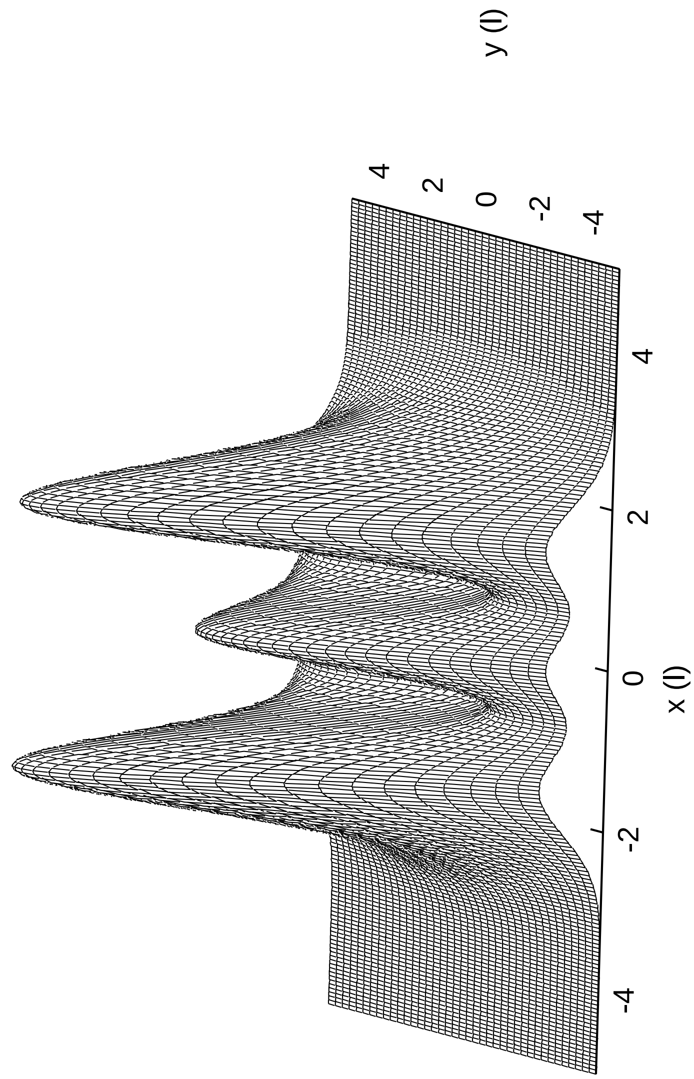

FIG. 1. The probability density for the state $N = 2$; $|\tanh \Phi_2| = 0.1, \Phi_1 = 0$ located at the origin $\boldsymbol{R} = 0$.

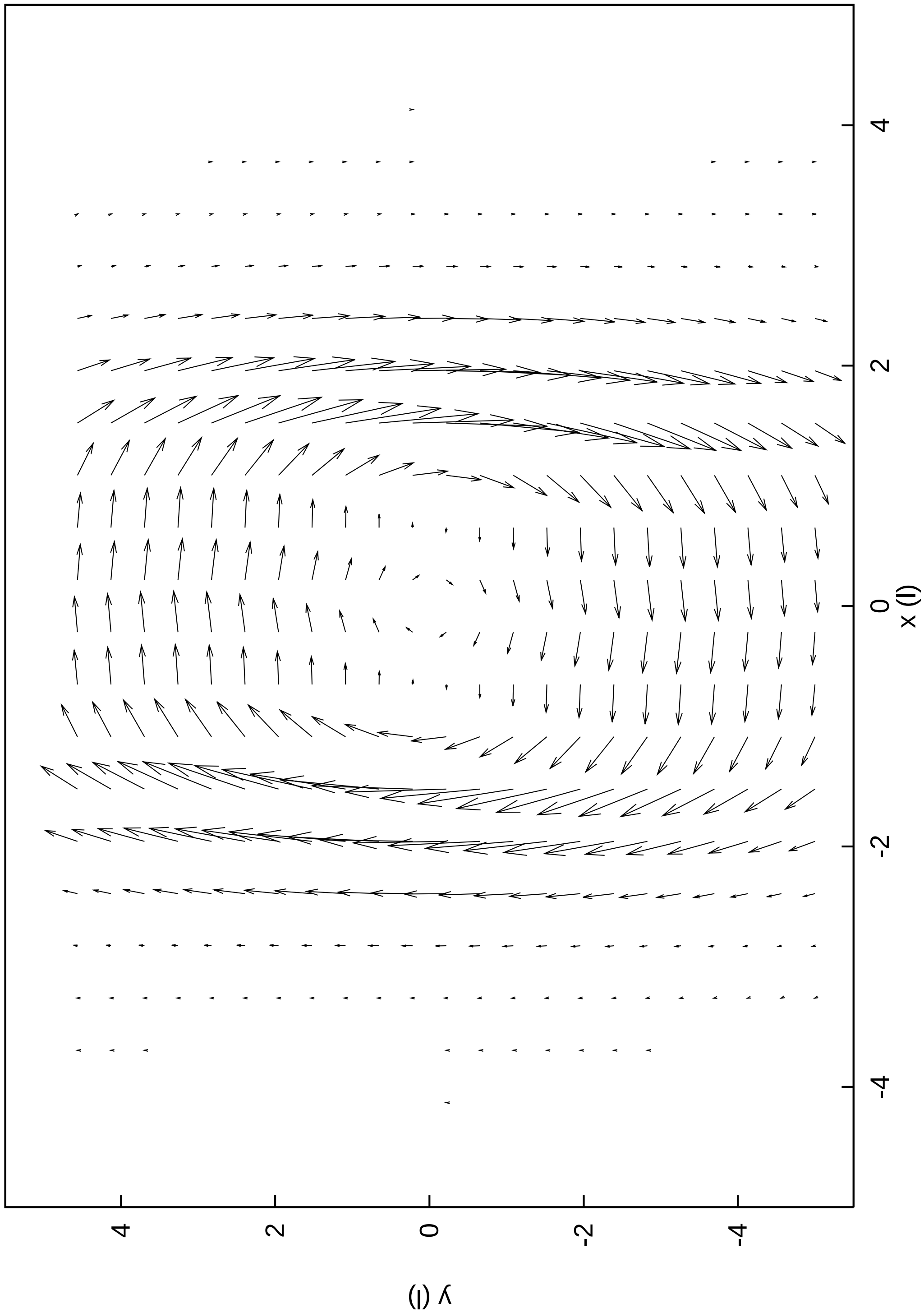

FIG. 2. The current density for the state $N = 2$; $|\tanh \Phi_2| = 0.1, \Phi_1 = 0$ located at the origin $\boldsymbol{R} = 0$.

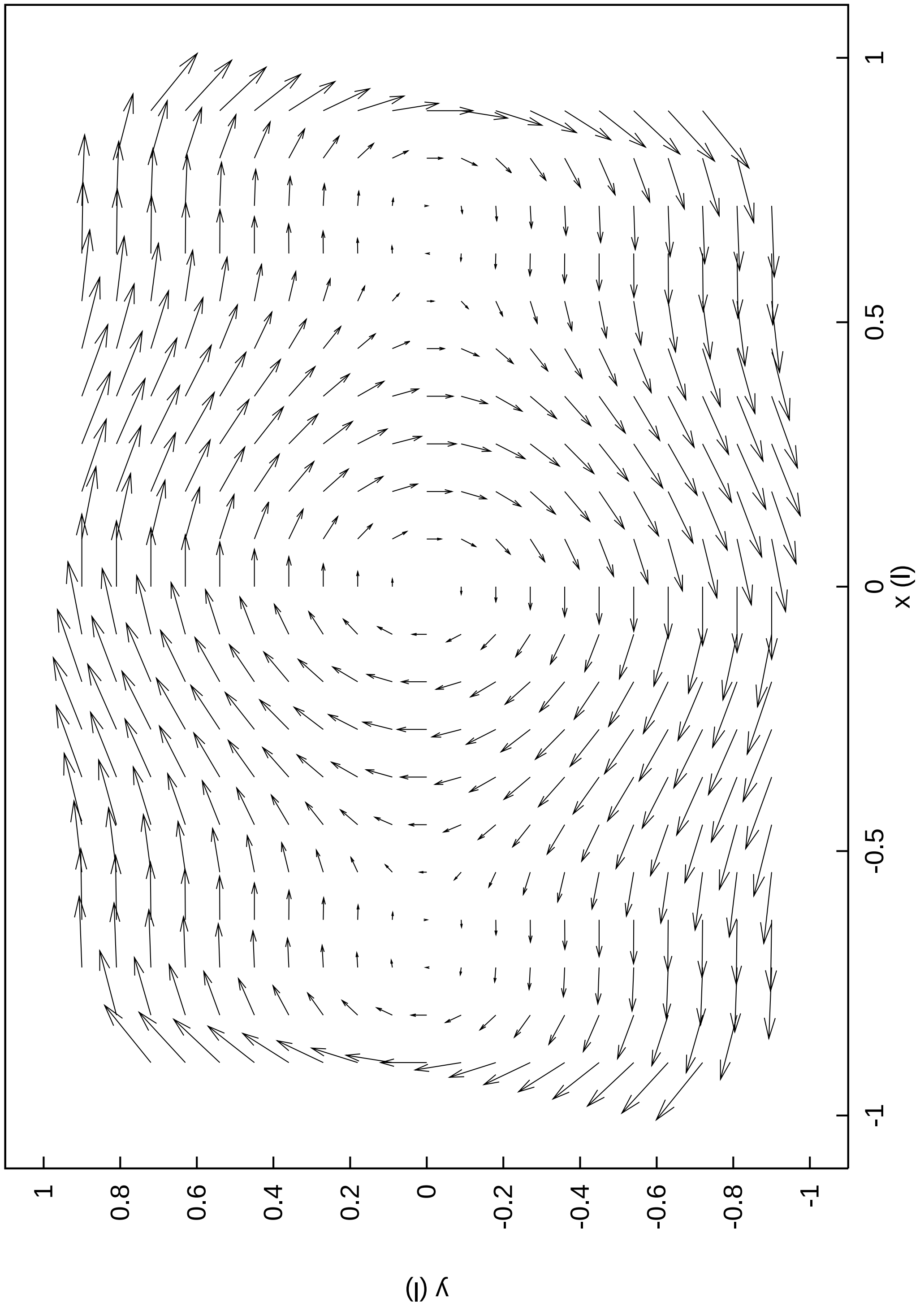

FIG. 3. Detail of the current density for the state $N = 2$; $|\tanh \Phi_2| = 0.1, \Phi_1 = 0$ located at the origin $\boldsymbol{R} = 0$.

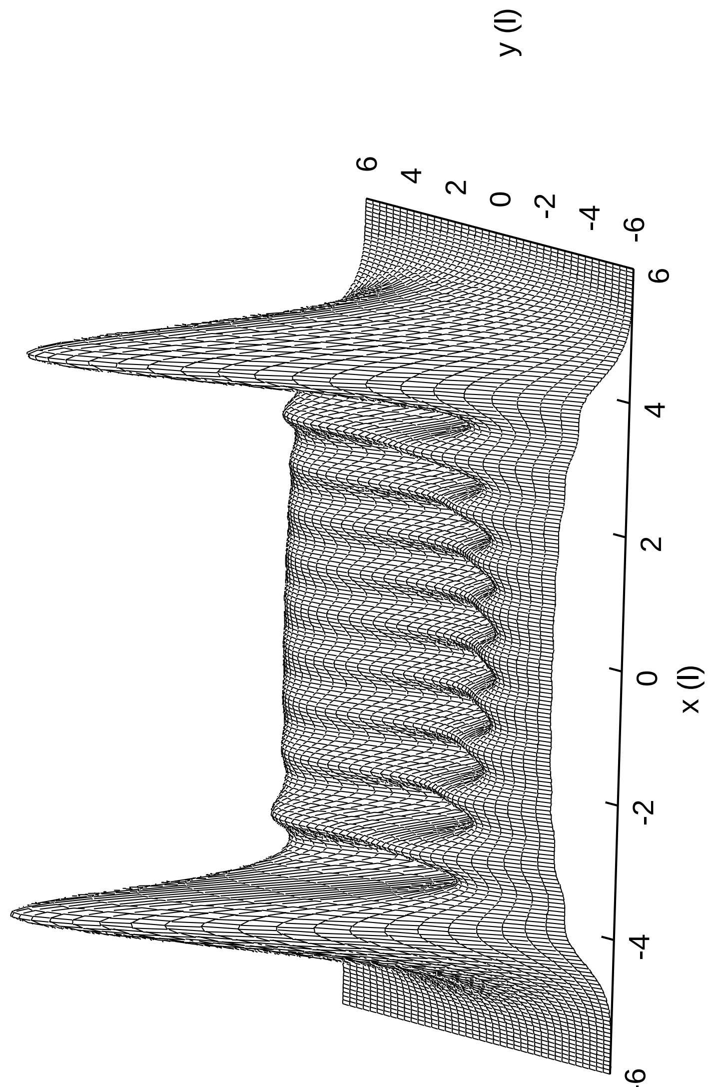

FIG. 4. The probability density for the state $N = 10$; $|\tanh \Phi_2| = 0.1, \Phi_1 = 0$ located at the origin $\boldsymbol{R} = 0$.